A first principles explanation of chirp modes

V Guruprasad, Inspired Research

In the Fourier spectrum $\int F(\omega)e^{i\omega(t-r/c)}\,d\omega$ of a d'Alembert solution $f(t-r/c)$, each component $F(\omega)e^{i\omega(t-r/c)}$ represents a spectral component $F(\omega)e^{i\omega t}$ arriving from the source, and the term r/c accounts for the travel delay. Consider next that a short interval δt after the initial analysis that yielded the Fourier coefficients $F(\omega)$, the labels on the receiver's frequency scale are shifted by $\delta\omega$. The signal $f(t-r/c)$ must then appear as $F(\omega+\delta\omega)e^{i\omega(t-r/c)}$ or equivalently, $F(\omega)e^{i(\omega-\delta\omega)(t-r/c)}$, against the receiver's shifted scale. If we kept shifting the receiver's scale discretely in-between spectral analyses, at each step the spectrum would be Fourier, with shifts equal to the cumulative shift of the scale.

However, if the receiver's scale changed *continuously*, during each frame of the spectral decomposition, at a fractional rate $\beta = \omega^{-1}d\omega/dt$, the components become $F(\omega)\exp(i\omega_0 e^{\beta t}t)$, or $F(\omega)\exp(i\omega_0 e^{\beta[t-r/c]}[t-r/c])$ including the travel delay. *A continuous shift of scale changes the integral transform itself, leading to chirps,* just as velocities result from the continuity of steps in Zeno's paradox. The kernel of the Fourier transform becomes $\exp(i\omega_0 e^{\beta t})$, for the same reason that it became $\exp(i[\omega-\delta\omega])$ with the discrete scale shifts. The decomposition is still strictly a receiver computation. *The very form of the chirp components implies the frequency lags* $\Delta\omega = \omega(e^{\beta[t-r/c]}-1) \approx -\omega\beta r/c$, *and fractional shifts* $z \equiv \Delta\omega/\omega = -\beta r/c$ between the received and the current source spectra. These lags cannot be the same for sources at different ranges from the receiver, *unless* $\beta = 0$, at which limit the chirp modes reduce to sinusoids.

The paradox that the distance manifests from a mere change of representation can only mean that *the information was always present but merely inaccessible via Fourier theory.* Invariance with respect to time t is the premise of Fourier's method, first applied to the heat equation. The additional constancy in $(t-r/c)$ of d'Alembert solutions implies invariance in r. However, d'Alembert concerns only $(t-r/c)$ – each chirp component above has continuous second order partial derivatives in t and r, and **trivially** satisfies d'Alembert's 1-d equation. Fourier was inspired by d'Alembert (and Euler), 25 years after the latter's death.

Radar chirps, like those emitted by bats and dolphins for echo ranging, are identical to a first order, and bear the same frequency lags in FMCW radars. It is often argued to me that FMCW doesn't fit since it must "mix" the echoes with the transmitter to get the range. "Mixing" is merely how difference frequencies, like IF in superheterodyne receivers, and the difference is the frequency lag. One cannot argue that the information of range comes from the transmitter signal, for that would be silly, but without such an argument, the analogy stands.

Another unfortunate fallacy is that the conservation of phase in d'Alembert solutions prohibits range information besides amplitude decay or parallax. The frequency lags are orthogonal to source information represented by amplitude and phase spectra, as shown by the universe itself as Hubble shifts. These have the same form, so the difference is only in how they are obtained.